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PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH

TIME LAG CONTAINING A SMALL PARAMETER

Carlos Perelló *

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By Carlos Perelló

Introduction.

In this paper we show that the method of Cesari and Hale for the determination of periodic solutions of ordinary differential equations can be extended to the case in which these equations contain a time lag.

An ordinary differential equation with time lag (also called functional differential equation) differs from those without lag in that the derivative of a solution function at a time t depends also on the values of this solution at times preceding t. We further restrict our equations by considering only time lags less than a fixed number r.

In the notation introduced by Hale [1] we consider equations of the form

$$(\alpha) \qquad \dot{x}(t) = F(t, x_t),$$

where F denotes a functional (real or complex) defined for each t and for the "segment of solution" x_t, of length r, preceeding t. Here x denotes an n-vector.

As a particular case we encounter the difference-differential equations

$$(\beta) \qquad \dot{x}(t) = f(t, x(t), x(t-\tau_1),\dots,x(t-\tau_\ell)).$$

We will consider here equations of the form

$$(\gamma) \qquad \dot{x}(t) = L(x_t) + N(t, x_t, \mu),$$

where L is linear in x_t (in a space to be defined) and $N(t, \varphi, \mu)$ tends to zero as both φ and the parameter μ tend to zero.

In analogy with what has been done for ordinary differential equations (see Hale [2]), we seek a method to determine the T-periodic solutions of (γ) when N is T-periodic in t.

Many of the methods which have been considered for ordinary differential equations are difficult to apply in the case of time lag as will be shown in the next paragraphs.

If in (α) F is T-periodic in t, we might assume that our solution has a trigonometric Fourier expansion of period T. We then reduce the problem of finding a T-periodic solution of (α) to that of solving the infinite number of equations obtained by equating coefficients. Making the above reduction and solving the equations which result is in general extremely difficult even when there is no lag present.

Cesari [3] shows that for nonlinear equations without lag it is not necessary to consider an infinity of solutions, but merely to see if some elements of a family of periodic functions, which are obtained as fixed points of a family of operators, satisfy a finite number ($2m+1$) of "determining equations". Any of these fixed points which satisfies the determining equations is a periodic solution. The difficulty lies in finding the fixed points and verifying that they satisfy the determining equations. By means of an implicit function theorem, however, he succeeds in showing that under certain circumstances it is sufficient to consider the ($2m+1$)-parameter family of trigonometric polynomial containing the first m harmonics instead of the family of fixed points. The determining equations can then actually be used to calculate the $2m+1$ coefficients of the polynomials satisfying them. Further it can be shown that the functions thus obtained lie in the vicinity of the periodic solutions of the equation. This is nothing more than the justification of the Galerkin procedure. The method is still very difficult to apply, even in the most simple cases.

The generalization of the above method to equations with lag will not be attempted here and will be the subject of some further publication. Let us remark, however, that the method of Cesari in [3] relies on the use of L^2 spaces and these do not seem the most appropriate for equations like (β), which we want to be able to include in our theory. It looks as if the modification of the method introduced by Knoblock [4] using uniform norms would generalize without trouble to lag equations including the difference-differential type.

The basis of the perturbation procedure of Cesari and Hale for (γ) without lag, as it is shown in the last part of [3], is essentially the same as in the case above. Now, however, we look for periodic solutions of the perturbed system which tend to periodic solutions of the linear system as the parameter μ tends to zero.

The generalization of this procedure to lag equations is made possible by decomposing equation (γ) by means of the projection operators defined by Hale [1]. We then obtain an ordinary differential equation without time lag perturbed with a term containing some lag element which couples this equation with a second one. By neglecting this lag element we obtain an ordinary perturbation problem which can be dealt with by the methods mentioned above. For small μ the periodic solutions of the unperturbed equation yielded by the determining equations are close to periodic solutions of equation (γ). In a first approximation we want to find the periodic solution of the linear equation to which the periodic solutions of the perturbed one tend.

The basic idea behind the decomposition in [1] and the reduction of the problem to equations without lag is to consider a function space as our phase space. Notice that the initial value problem for equation (α) is well posed if we give as initial condition a function defined in an interval of length r. In fact there might be an infinity of functions which satisfy the equation and pass through a given point of the n-dimensional euclidean space.

In section I we give the required background material on equations with time lag. In section II we develop the method for (γ) nonautonomous. The reduction of (γ) autonomous to the previous case is treated in section III. In the next section we show how the basic procedure can be used to determine the asymptotic stability of a periodic solution. In order to do that we require some simple results from the theory of periodic linear equations with time lag that we borrow from Stokes [5] and Shimanov [6]. In section V we present a simple example arising from a control system with a delay in the feedback. Section VI is devoted to the procedure to be followed when we have to use higher order terms to ascertain the existence of periodic solutions and an example of the application of this procedure is given.

I. Preliminaries.

Let E^n be the n-dimensional complex euclidean space and consider the continuous function $x : [-r, \tau) \to E^n$, τ, $r > 0$. Consider also the space $C([-r, 0], E^n) = C$ of the continuous functions defined in the closed interval $[-r, 0]$ with range in E^n, with the sup norm. We define the operator $\mathscr{A}_t$ associating an element of C to x for every t in $[0, \tau)$ by means of the rule

$$\mathscr{A}_t(x) = x(t + \theta), \quad \theta \text{ in } [-r, 0].$$

In order to simplify the notation we shall use $\mathscr{A}_t(x) = x_t$. (See Hale [1]).

Given a functional $F : R \times C \to E^n$ an letting $\dot{x}(t)$ represent the right hand derivative of x at t, we define a functional-differential equation as the relation

$$\dot{x}(t) = F(t, x_t). \tag{1}$$

The function F does not need to be defined on the whole of $R \times C$. In fact for our use in this work we shall suppose it defined for all R and in an open ball $C_H = \{\varphi \in C : \|\varphi\| < H\}$.

We say that $x(\sigma, \varphi) : [\sigma - r, \tau) \to E^n$ is a solution of (1) with initial value φ at σ if there exists $\tau > \sigma$ such that $x_t(\sigma, \varphi)$ is in C_H for t in $[\sigma, \tau)$, $x_\sigma(\sigma, \varphi) = \varphi$ and (1) is satisfied by $x(\sigma, \varphi)(t)$, t in $[\sigma, \tau)$.

If (1) is autonomous, i.e., F does not depend explicitly on t, and we choose $\sigma = 0$, we abbreviate $x_t(\sigma, \varphi)$ by $x_t(\varphi)$.

Consider the case in which (1) is autonomous and F is a continuous linear functional:

(2) $$\dot{u}(t) = L(u_t).$$

This case is particularly important to us, since most of the properties of our perturbed equations (γ) depend on the unperturbed ones.

In the next paragraphs we summarize the parts of the theory of (2) which are relevant to this work. For a more detailed exposition, with proofs, see [1].

The Riesz representation theorem tells us that we can write

(3) $$L(\varphi) = \int_{-r}^{0} [d\eta(\theta)]\varphi(\theta) ,$$

where $\eta(\theta)$ is an $n \times n$ matrix of function of bounded variation on $[-r, 0]$. On the other hand it is well known that (2) has a unique solution defined for t in $[0, \infty)$ for any initial value φ in C at zero (see Krasovskii [7], or Halanay [8]).

We define the semi-group of operator $U(t) : C \to C$ by

$$U(t)\varphi = u_t(\varphi) ,$$

where $u(\varphi)$ is the solution of (2) with initial value φ at zero. For each $t > 0, \tau > 0$, $U(t)$ is a bounded linear operator satisfying $U(t + \tau) = U(t)U(\tau)$.

In terms of the matrix η appearing on (3) we find that the characteristic values of (2) are given as the roots of the characteristic equation

(4) $$\det (\lambda I - \int_{-r}^{0} [d\eta(\theta)]e^{\lambda\theta}) = 0$$

There are only a finite number of roots of (4) in any half plane $\operatorname{Re} z \geq \gamma$, and each of these roots has finite multiplicity.

If λ has multiplicity k, then there are k, and no more than k linear independent solutions of (2) of the form $y(t) = p(t)e^{\lambda t}$, where $p(t)$ is a polynomial with coefficients in E^n of degree $\leq k-1$.

We observe that these solutions can be prolongated backwards, i.e., there is a function $y\colon R \to E^n$, such that

$$u(y_\tau)(t) = y(t-\tau) \quad \text{for} \quad t,\ \tau \in R.$$

Let Y denote the matrix having as columns the k linealy independent solution mentioned above. Then there exists a constant matrix B, with λ as only characteristic value, such that

$$Y(t) = Y(0)e^{Bt} \quad , \quad t \in R.$$

If we define $\Phi = Y_0$, i.e., the matrix whose columns are the elements of C corresponding to y_0, then we have:

$$Y_t = U(t)\Phi = \Phi e^{Bt},\ \Phi(\theta) = \Phi(0)e^{B\theta},\ \theta \in [-r,\ 0].$$

This relation says that Φ is the basis of a finite dimensional subspace $P(\Lambda)$ of C which is invariant under $U(t)$. In this subspace we can extend the definition of $U(t)$ to negative values of t by taking $U(-t) = \Phi\ e^{-Bt}$.

Given any finite set $\Lambda = \{\lambda_i\}$ of characteristic values of (2) it is possible to obtain a set of functions of the form $y(t) = p(t)e^{\lambda_i t}$, $t \in R$, such that, if Y denotes the matrix whose columns are this basis, there exists a constant matrix B such that

$$Y(t) = Y(0)e^{Bt}\ , \quad t \in R, \tag{5}$$

where B has as characteristic values the elements of Λ.

The linear subspace spanned by the columns of $Y_o = \Phi$ is called the <u>generalized</u> <u>eigenspace</u> associated with Λ, and will be denoted by $P(\Lambda)$.

If φ is an element of $P(\Lambda)$ we have then

$$u_t(\varphi) = U(t)\varphi = \Phi e^{Bt} b\ , \quad \varphi = \Phi\, b\ . \tag{6}$$

That shows that in $P(\Lambda)$ the behavior of the solution is the same as that of an ordinary differential equation with constant coefficients.

If L is a real functional $(L : C \to R^n)$, and we are only interested in the real part of u_t, then we know that both λ and $\bar{\lambda}$ are characteristic roots. By associating λ with $\bar{\lambda}$ we can choose Φ as a matrix whose elements are real functions and such that their columns form a basis for the real part of $R(\Lambda)$. In this case B will be a real constant matrix.

We will next characterize the space $Q(\Lambda)$ complementary to $P(\Lambda)$ which will be also invariant under the operator $U(t)$ for $t \geq 0$. Every element φ of C will then be uniquely expressible as the sum of an element of $P(\Lambda)$ and one of $Q(\Lambda)$ These elements are called the <u>projections</u> of φ on $P(\Lambda)$ and $Q(\Lambda)$ respectively. If p_P and p_Q designate the operators of projection we can write

$$\varphi = p_P(\varphi) + p_Q(\varphi)\ . \tag{7}$$

To abbreviate we designate $p_P(\varphi)$ and $p_Q(\varphi)$ by φ^P and φ^Q respectively. We write then (7) as

$$\varphi = \varphi^P + \varphi^Q\ .$$

We obtain the characterization of $Q(\Lambda)$ with the help of the following equation, known as the adjoint to (2)

(8) $$\dot{v}(s) = -\int_{-r}^{0}[d\eta^T(\theta)]v(s-\theta)\ , \quad s \leq 0\ ,$$

(η^T is the transpose of η), and its associated charactertistic equation

(9) $$\det\ (\lambda I - \int_{-r}^{0}[d\eta^T(\theta)]e^{\lambda\theta}) = 0\ .$$

The solutions of (4) and (9) are the same. A solution of (8) is uniquely determined by giving an initial condition ψ in $C([0,\ r],E^n) \overset{\text{def}}{=} C^*$ at 0, and integrating (8) for $s \leq 0$. To any φ in C and ψ in C^* we associate the bilinear form $(\psi,\ \varphi)$ defined by

(10) $$(\psi,\ \varphi) = \psi^T(0)\varphi(0) - \int_{-r}^{0}\int_{0}^{\theta}\psi^T(\xi - \theta)[d\eta(\theta)]\varphi(\xi)d\xi\ .$$

If Φ is a basis for $P(\Lambda)$ and Ψ is a basis for $P^*(\Lambda)$ (the generalized eigenspace of Λ in C^*), then $(\Psi,\ \Phi) = (\psi_j,\ \varphi_k))$ is non singular and, by changing the bases, can be taken as the identity matrix. Let us then assume

(11) $$(\Psi,\ \Phi) = I\ .$$

The space $Q(\Lambda)$ is characterized by

(12) $$Q(\Lambda) = \{\varphi \in c : (\Psi,\ \varphi) = 0\}.$$

If $\varphi \in Q(\Lambda)$, then $U(t)\varphi \in Q(\Lambda)$ for $t \geq 0$. In this case the solutions are not necessarily defined for negative t as in $P(\Lambda)$.

We have then that the projection operator p_P is defined by

$$\varphi^P = p_P\varphi = \Phi(\Psi,\ \varphi)$$

and

$$\varphi^Q = p_Q\varphi = \varphi - p_P\varphi \ .$$

Consider now the equation

$$\dot{x}(t) = L(x_t) + N(t, x_t). \tag{13}$$

We want an expression, similar to the variation of parameters formula, which will give the solution of (13) for a given initial value in terms of the solutions of (2).

Let $X(t)$ be the $n \times n$ matrix whose columns are the solutions for $t \geq -r$ of equation (2) with $X(t) = 0$ for t in $[-r, 0)$ and $X(0) = I$, the identity matrix. Then we have the following representation for the solutions of (13) with initial value φ at σ (see Halanay [8],[9] and Hale-Perelló [10]):

$$\begin{aligned} x(t) &= U(t-\sigma)\varphi(0) + \int_\sigma^t X(t-\tau)N(\tau, x_\tau)d\tau, && t \geq 0, \\ x(\sigma + \theta) &= \varphi(\theta) && \theta \epsilon [-r, 0] \end{aligned} \tag{14}$$

It is shown in [9] that by projecting X_0 on P and Q as indicated previously, that is, by taking

$$X_0^P = \Phi(\Psi, X_0) = \Phi\Psi^T(0)$$

$$X_0^Q = X_0 - X_0^P \quad ,$$

the equation (14) can be decomposed as follows:

$$x_t^P(\theta) = U(t-\sigma)\varphi^P(\theta) + \int_\sigma^t U(t-\tau)X_0^P(\theta)N(\tau, x_\tau)d\tau\ , \quad t \in R$$

(15)

$$x_t^Q(\theta) = U(t-\sigma)\varphi^Q(\theta) + \int_\sigma^t U(t-\tau)X_0^Q(\theta)N(\tau, x_\tau)d\tau\ , \quad t \geq 0$$

From now on, in order to abbreviate, we will not write the θ when using these formulas.

II. The nonautonomous equation

Consider equation (2) and assume that Λ is the set of all of its characteristic roots of the form $i\frac{2\pi n}{T}$, n integer. We know there is only a finite number of such roots. Assume, moreover that the dimension of the eigenspaces spanned by these roots coincide with their multiplicity. Then $P(\Lambda)$ will consist of all those functions which are initial values of T-periodic solutions of (2).

According to (6) the orbits (or paths) of the equation in $P(\Lambda)$ are given by $u_t(\varphi) = \Phi e^{Bt}b$, where $\varphi = \Phi b$ and B is a $p \times p$ matrix which has the elements of Λ as eigenvalues and has simple elementary divisors. Notice that p and n are not related, and any can be larger than the other. If $w(t) = (\Psi, w_t)$ we have that for u_t in $P(\Lambda)$, $w(t)$ satisfies the linear equation

$$\dot{w}(t) = Bw(t). \tag{16}$$

We introduce some more notation that we need in the next pages:

S^P denotes the space of T-periodic functions y from R into E^P with the norm $\|y\|_s = \sup\{|y(t)|,\ t \in R\}, |y|^2 = y^*y$, y^* the conjugate transpose of y.

Σ denotes the space of T-periodic functions x_t from R into C with the norm $\|x_t\| = \sup\{\|x_t\|,\ t \in R\}$, $\|x_t\| = \sup\{|x(\theta)|,\ \theta \in [-r, 0]\}, |x|$ as above.

$\mathscr{O}: S^p \to S^p$ denotes the operator defined by

$$\mathscr{O}(f) = \frac{1}{T}\int_0^T e^{B(t-\tau)} f(\tau)\,d\tau.$$

Notice that $\mathscr{O}(f)$ is of the form $e^{Bt}a$ and hence will correspond to some solution of (16).

By $\Omega : \Sigma \to \Sigma$ we denote the operator definded by

$$\Omega(x_t) = \Phi\, \mathscr{O}(\psi, x_t) .$$

Here we are using the notation $x_t : R \to C$ even if there is no $x : R \to E^n$ corresponding to it (see the definition of x_t at the beginning of section I). The use of this notation is similar to the abuse made when we write $x(t): R \to E^n$ which we do very frequently in order to use less symbols.

To begin with we will find necessary and sufficient conditions for the equation

$$\dot{x}(t) = L(x_t) + f(t) , \tag{17}$$

with f in S^n and L as above to have T-periodic solutions. Such conditions are given in a more general theorem in [8], but we prefer to include the proof for our case which is much simpler.

<u>Lemma 1.</u>

<u>If</u> $f \in S^p$, <u>then the equation</u>

$$\dot{y}(t) = B\, y(t) + f(t) , \tag{18}$$

B <u>as in (16), has a periodic solution if and only if</u> $\mathscr{O}(f) = 0$, <u>and in this case for every</u> $a \in E^p$ <u>there is a unique solution</u> $y^*(a)$ <u>of (18) such that</u> $\mathscr{O}(y^*(a) = e^{Bt}a = w(a)(t)$, i.e., $\mathscr{O}(y^*(a))$ <u>is the solution of (16) with initial value</u> a <u>at</u> $t = 0$.

<u>Moreover the following estimate holds</u>

$$\|y^*(a) - w(a)\|_S \leq K \int_0^T |f(\tau)|\, d\tau ,$$

<u>where</u> K <u>does not depend on</u> f <u>or</u> a.

Remark: $y^*(a)$ is not necessarily the solution of (18) with initial value a at $t = 0$.

<u>Proof</u>: The solution of (18) with initial value y_o at $t = 0$ is given by

$$y(t) = e^{Bt}y_o + \int_o^t e^{B(t-\tau)}f(\tau)d\tau \ . \tag{19}$$

As $e^{Bt}y_o$ is T-periodic, in order to have $y(t)$ T-periodic it is necessary and sufficient that $\int_o^t e^{B(t-\tau)}f(\tau)d\tau$ be T-periodic, that is, we require the $\int_o^T e^{-B\tau}f(\tau)d\tau = 0$ or, using our notation, $\mathcal{O}(f) = 0$.

From (19) we have for $y \in S^p$ that $e^{-Bt}y(t) = a + g(t)$, where $a = y_o + \frac{1}{T}\int_o^T \int_o^t e^{-B\tau}f(\tau)d\tau = y_o + c$, and g is a function in S^p with mean value 0.

Applying the operator $\mathcal{O}$ to $y \in S^p$ we obtain

$$\mathcal{O}(y)(t) = e^{Bt}(y_o + \frac{1}{T}\int_o^T \int_o^\xi e^{-B\tau}f(\tau)d\tau \ d\ \xi) = e^{Bt}a = v(a)(t).$$

Hence $\mathcal{O}$ gives a 1-1 correspondence between the periodic solutions of (18) and those of (16).

From the fact that

$$\|g\|_S \leq 2T \ \| \ e^{-Bt}\|_S \int_o^t |f(\tau)|d\tau = k \int_o^t |f(\tau)|d\tau$$

the last part of the lemma follows by taking $K = \|e^{Bt}\|_S k$. For the matrices e^{Bt} and e^{-Bt} we are using as S norm the supremum of the square root of the sum of the product of their elements by their conjugates for all t.

<u>Lemma 2.</u>

<u>If</u> h <u>is in</u> S^n, <u>then there exists a unique</u> $y \in Q(\Lambda)$ <u>such that</u>

(20) $$x_t^{*Q} = U(t)\varphi + \int_o^t U(t-\tau)X_o^Q\, h(\tau)d\tau$$

is T-periodic.

Moreover we have

$$\|x_t^{*Q}\|_\Sigma \leq K'\int_0^T |h(\tau)|d\tau \quad ,$$

where K' *is independent of the* h *chosen.*

Proof: If x_t^{*Q} is T-periodic we have

$$\varphi = U(T)\varphi + \int_0^T U(T-\tau)X_0^Q\, h(\tau)d\tau, \quad \text{that is}$$

$$\varphi = (I-U(T))^{-1}\int_0^T U(T-\tau)X_0^Q\, h(\tau)d\tau \ .$$

We have that $I-U(T)$ has an inverse if $(I-U(T))\varphi = 0$ implies $\varphi = 0$. This is the case, since we have assumed that there are no T-periodic solutions of (2) in $Q(\Lambda)$ besides the identically zero. Hence φ is uniquely determined. Notice that $\int_0^T U(T-\tau)X_0^Q(\theta)h(\tau)d\tau$ is a continuous function in θ for θ in $[-r, o]$.

The expression for x_t^{*Q} is

$$x_t^{*Q} = (I-U(T))^{-1}\int_t^{t+T} U(t+T-\tau)X_0^Q\, h(\tau)d\tau \ .$$

The estimate on the Σ-norm of x_t^{*Q} is obtained as follows:

$$\|x_t^{*Q}\|_\Sigma = \|(I-U(T))^{-1}\| \sup_{\tau\in[t,t+T]} \|U(t+T-\tau)X_0^Q\| \int_0^T |h(\tau)|d\tau =$$

$$= \|(I-U(T))^{-1}\| \sup_{t\in[0,\ T]} \|U(t)X_0^Q\| \int_0^T |h(\tau)|d\tau = K' \int_0^T |h(\tau)|d\tau \ .$$

By using our decomposition (15) we obtain immediately the desired property concerning equation (17):

Theorem 1.

The equation

$$\dot{x}(t) = L(x_t) + f(t) \tag{17}$$

with $f \in S^n$ *and* L *as in* (2) *has a* T*-periodic solution if and only if* $\mathcal{O}(\Psi^T(0)f) = 0$, *and in this case, for every* Φa *in* $P(\Lambda)$ *there exists a unique solution* $x_t^*(a)$ *such that* $\mathcal{O}(\Psi, x_t(a)) = e^{Bt}a$.

Moreover the following estimate holds:

$$\|x_t^*(a) - u_t(\Phi a)\|_\Sigma \leq K'' \int_0^T |f(\tau)| d\tau \quad , \tag{21}$$

where K'' *does not depend on* f.

Notice that the condition $\mathcal{O}(\Psi^T(0)f)$ is equivalent to

$$\begin{aligned} &\int_0^T e^{-B\tau}\Psi^T(0)f(\tau)d\tau = 0 \ , \quad \text{or} \\ &\int_0^T \Psi^T(\tau)f(\tau)d\tau = 0 \ , \end{aligned} \tag{22}$$

that is, in order for (17) to have some T-periodic solution it is neccessary and sufficient that f be orthogonal, in the sense of (22), to the T-periodic solutions $\Psi^T(t)$ of the equation adjoint to (2) (See [8]),

In the case in which (2) has no T-periodic solutions besides the identically zero, then there is a unique T-periodic solution for every f in S^p.

The following two lemmas follow trivially from the ones above, but we prefer to state them explicitly for easier reference.

Lemma 3.

If x_t is an element of Σ, then for every $a \in E^p$ the equation

$$\dot{y}(t) = B\, y(t) + \Psi^T(0)N(t, x_t) - \mathcal{O}(\Psi^T(0)N(t, x_t)) , \tag{23}$$

where $N(t, \varphi)$ is a functional of period T in t, continuous with respect to (t, φ) and uniformly lipschitzian in φ in C_H, has a unique solution $y^*(a, x_t) \in S^p$ such that $\mathcal{O}(y^*(x_t)) = e^{Bt}a = w(a)(t)$.

To abbreviate we are going to write

$$e^{-Bt}(\Psi^T(0)N(t, x_t) - \mathcal{O}(\Psi^T(0)N(t, x_t)) = f(x_t)(t)$$

With this notation we have for the solution $y^*(a, x_t)$ of (23):

$$y^*(a, x_t)(t) = e^{Bt}\left(a + \int_0^t f(x_t)(\tau)d\tau - \frac{1}{T}\int_0^T\int_0^\sigma f(x_t)(\tau)d\tau\, d\sigma\right) = \tag{24}$$

$$= e^{Bt}(a + g(t))$$

Here $g(t)$ stands for the unique T-periodic function with zero mean value whose derivative is $f(x_t)(t)$.

If we want to express $g(t)$ as an integral we have to deal with its components separately. In fact if the components of g are complex we have to deal separately with the real and imaginary part for each component. We can choose ξ_i , η_i in $[0, T]$, $i = 1,\ldots,p$, such that $\operatorname{Re} g_i(\xi_i) = \operatorname{Im} g_i(\eta_i) = 0$. We will have then that

$$\operatorname{Re} g_i = \int_{\xi_i}^t f(x_t)(\tau)d\tau \quad\text{and}\quad \operatorname{Im} g_i(t) = \int_{\eta_i}^t f(x_t)(\tau)d\tau \tag{25}$$

have zero mean value. If $\overline{\xi}$ denotes the vector of E^p with components $(\zeta_1,\ldots,\zeta_p)$, $\zeta_i = \xi_i + i\eta_i$, we will write

$$g(t) = \int_{\overline{\zeta}(x_t)}^{t} f(x_t)(\tau)d\tau \tag{26}$$

for the vector function with components (25). Notice that $\overline{\zeta}(x_t)$ is not necessarily uniquely determined.

Observe that if we take a new $x_t' \in \Sigma$, the following linear property holds for some $\overline{\zeta}(x_t + x_t')$ with components with real and imaginary parts in $[0, T]$:

$$\int_{\overline{\zeta}(x_t)}^{t} f(x_t)(\tau)d\tau + \int_{\overline{\zeta}(x_t')}^{t} f(x_t')(\tau)d\tau = \int_{\overline{\zeta}(x_t+x_t')}^{t} f(x_t')(\tau) + f(x_t)(\tau))d\tau. \tag{27}$$

This follows because both terms of the first member have mean value zero and so must have their sum, $h(t)$ say. On the other hand $h'(t) = f(x_t)(t) + f(x_t')(t)$ and there exists $\overline{\zeta}(x_t + x_t')$ in $[0, T]$ such that the second member of (27) is equal to $h(t)$.

Lemma 4.

If $x_t \in \Sigma$, then under the same hypothesis as above, there exists a unique φ in $Q(\Lambda)$ such that

$$V(t)\varphi + \int_0^t U(t-\tau)X_0^Q\, N(\tau, x_\tau)d\tau$$

is in Σ.

The main purpose of this section is to give conditions under which the following equation has T-periodic solutions:

$$\dot{x}(t) = L(x_t) + N(t, x_t, \mu). \tag{28}$$

Here L is as before and $N(t, \varphi, \mu)$ fulfills the following conditions in the region $R \times C_H \times [-\mu_o, \mu_o]$ for some $H, \mu_o > 0$:

i) $N(t, \varphi, \mu)$ is continuous in (t, φ, μ), $N(t, 0, 0) = 0$,

ii) $N(t, \varphi, \mu)$ is T-periodic in t,

iii) $|N(t, \varphi, \mu) - N(t, \varphi_2, \mu)| \leqq \eta(|\mu|, H) \|\varphi_1 - \varphi_2\|$,

φ_1, φ_2 in C_H for some continuous function η defined in $[0,\mu_o] \times [0, H_o]$, η nondecreasing in $|\mu|$ and H and $\eta(0, 0) = 0$.

The above conditions are enough to insure locally the existence and uniqueness of solution for any μ in $[-\mu_o, \mu_o]$ and any initial condition φ in C_H at a time σ in R. If we do not leave C_H for any t, then the solution is defined for all $t \geqq \sigma$, and if for some φ we have that $x_T(\varphi) = \varphi$, $x_t(\varphi) \in C_H$ for $t \in [0, T]$, then we can take $x_t(\varphi)$ T-periodic for every t in R. Notice here that it may happen that there is no uniqueness of solution going backwards in time. It may occur that two solutions with different initial conditions at σ coincide after some $t > \sigma$. For instance the equation $\dot{x} = Ax$ considered as a lag equation in the phase space $C([-r, 0], E^n)$, $r > 0$, is such that any solution with initial condition φ such that $\varphi(0) = 0$ will be zero for $t \geqq 0$.

For any α, $0 < \alpha < 1$, and for any $a \in E^P$ fulfilling $\|\Phi\, e^{Bt}a\|_\Sigma \leqq \alpha H$, we denote by $\Sigma_{a,H}$ the following subset of Σ:

$$(29) \qquad \Sigma_{a,H} = \{x_t \in \Sigma : \Omega(x_t) = \Phi\, e^{Bt}a,\ \|\Omega(x_t)\|_\Sigma \leqq \alpha H,\ \|x_t\|_\Sigma \leqq H\} ,$$

i.e. the set of those T-periodic solutions from R into C which never leave the ball C_H and such that their "average" Ω equals $\Phi\, e^{Bt}a$ and is contained in the smaller ball $C_{\alpha H}$.

We do not make explicit the choice of α, but we have to keep in mind that its value is fixed throughout the whole reasoning. Notice also that if η is independent of H our results will be valid for any H if μ is small enough.

Lemma 5.

There exist $\mu_1 > 0$, $H > 0$ such that for every $a \in E^P$ with $\|\Phi e^{Bt} a\|_\Sigma \leqq \alpha H$ there exists a unique $x_t = x_t(a, \mu)$ in $\Sigma_{a,H}$ satisfying the relations

(30) $$\dot{y}(t) = By(t) + \Psi^T(0)\, N(t, x_t, \mu) - \mathscr{O}(\Psi^T(0) N(t, x_t, \mu)),$$

(31) $$x_t^Q = U(t)x_0^Q + \int_0^t U(t-\tau)X_0^Q\, N(\tau, x_\tau, \mu)d\tau$$

for every μ with $|\mu| \leqq \mu_1$. Furthermore this $x_t(a, \mu)$ is continuous on (a, μ).

Proof:

We use the notation

(32) $$n(x_t, \mu)(t) = e^{-Bt}(\Psi^T(0)N(t, x_t, \mu) - \mathscr{O}(\Psi^T(0)N(t, x_t, \mu)) ,$$

for the function of t which results from substuting a given x_t in Σ in the right hand side of (32).

If we take z_t in Σ and substitute it in (30) and (31) we obtain two uncoupled equations:

(33) $$\dot{y}(t) = By(t) + e^{Bt} n(z_t, \mu)(t)$$

$$x_t^Q = U(t)x_o^Q + \int_o^t U(t-\tau)X_o^Q N(\tau,\ z_\tau,\ \mu)d\tau\ , \tag{34}$$

According to Lemma 3, for any a in E^p equation (33) has a unique T-periodic solution given by

$$y^*(t,\ a,\ z_t,\ \mu) = e^{Bt}(a + \int_{\overline{\zeta}(z_t,\mu)}^{t} n(z_t,\ \mu)(\tau)d\tau)\ . \tag{35}$$

In a similar way, according to Lemma 4, equation (34) has a unique periodic solution x_t^Q given by

$$x_t^{*Q}(z_t,\ \mu) = (I-U(T))^{-1}\int_t^{t+T} U(t + T - \tau)X_o^Q N(\tau,\ z_\tau,\ \mu)d\tau. \tag{36}$$

Let's define the operator $\mathscr{F}(a,\ \mu)$ from Σ into Σ by

$$\begin{aligned}\mathscr{F}(a,\ \mu)(z_t) &= \Phi y^*(t,\ a,\ z_t,\ \mu) + x_t^{*Q}(z_t,\ \mu) = \\ &= \mathscr{F}^P(a,\ \mu)(z_t) + \mathscr{F}^Q(a,\ \mu)(z_t).\end{aligned} \tag{37}$$

We will show that for μ, H small enough $\mathscr{F}(a,\ \mu)$ maps $\Sigma_{a,H}$ into itself and that it is a contraction. Consequently there is a unique element in $\Sigma_{a,H}$ fixed under $\mathscr{F}(a,\ \mu)$.

The fact that $\Omega(\mathscr{F}(a,\ \mu)(z_t)) = \Phi\ e^{Bt}a$ is obvious. We have to show now that $\|\mathscr{F}(a,\ \mu)(z_t)\|_\Sigma \leq H$ if z_t is in $\Sigma_{a,H}$ and μ is sufficiently small.

From Lemma 1 we have the estimate

$$\|y^*(t,\ a,\ z_t,\ \mu) - e^{Bt}a\| \leq K \int_0^T |e^{B\tau} n(z_t,\ \mu)(\tau)|\, d\tau\ ,$$

for some K independent of n.

Since $\|\mathcal{O}(f)\|_s \leq \|f\|_s$, we get the estimate

$$\|\mathcal{F}^P(a,\ \mu)(z_t)\|_\Sigma = \|\Phi y^*(t,\ a,\ z_t,\ \mu)\|_\Sigma \leq$$

$$\leq \|\Phi\ e^{Bt}a\|_\Sigma + 2KT\ \|\ \Phi\ \|\|\Psi\|(\eta(|\mu|,H)H + \kappa(|\mu|)) \leq$$

$$\leq b + kK(\eta(|\mu|,H) + \kappa(|\mu|)) \leq$$

where κ is continous, increasing and $\kappa(0) = 0$.

By Lemma 2 we have

$$\|\mathcal{F}^Q(a,\ \mu)(z_t)\|_\Sigma \leq 2K'T\ \|\Psi\|(\eta(|\mu|,H)H + \kappa(|\mu|)) =$$

$$= k'K'(\eta(|\mu|,H)H + \kappa\ (|\mu|))\ .$$

It is sufficient to take

$$(\eta(|\mu|,H) + \kappa(|\mu|))(Kk + K'k') \leq H - b$$

to have $\|\mathcal{F}(a,\ \mu)(z_t)\|_\Sigma \leq H$ and hence $\mathcal{F}(a,\ \mu)(z_t)$ in $\Sigma_{a,H}$. Due to the continuity of η and κ we can choose $\mu_1' > 0$, $H_1 > 0$ such that $\eta(\mu_1',H_1)H_1+\kappa(\mu_1') \leq \dfrac{H_1(1 - \alpha)}{Kk + K'k'}$, and then $\mathcal{F}(a,\ \mu)$ maps Σ_{a,H_1} into Σ_{a,H_1} for $|\mu| \leq \mu_1'$.

We will now prove the contracting property of $\mathscr{F}(a, \mu)$, namely that for $|\mu|$ small enough then exists a $\delta_1 < 1$ such that for z_t and z_t' in $\Sigma_{a,H}$ the following holds:

(38) $$\|\mathscr{F}(a, \mu)(z_t) - \mathscr{F}(a, \mu)(z_t')\|_\Sigma \leqq \delta_1 \|z_t - z_t'\|_\Sigma \ .$$

According to (32) and (35) we have

$$\|\mathscr{F}^P(a, \mu)(z_t) - \mathscr{F}^P(a, \mu)(z_t')\|_\Sigma \leqq$$

$$\leqq \|\Phi\| \left\| \int_{\zeta(z_t - z_t')}^{t} (n(z_t, \mu)(\tau) - n(z_t', \mu)(\tau)) d\tau \right\|_s \leqq$$

$$\leqq 2 \|\Phi\| \ T \ \eta(|\mu|, H) \|\Psi\| \|z_t - z_t'\|_\Sigma = \eta(|\mu|) \ k \ \|z_t - z_t'\|_s \ .$$

Using (36) we get:

$$\|\mathscr{F}^Q(a, \mu)(z_t) - \mathscr{F}(a, \mu)(z_t')\| \leqq$$

$$\leqq K' \int_0^T |N(\tau, z_t, \mu) - N(\tau, z_t', \mu)| d\tau \leqq$$

$$\leqq K' T \ \eta(|\mu|, H) \|z_t - z_t'\|_\Sigma \ .$$

We can choose $\mu_1'' > 0$, $H_2 > 0$ such that $\eta(|\mu|, H_2)(k + K'T) < 1$ for $|\mu| \leqq \mu_1''$.

By choosing $\mu_1' = \min \{\mu_1', \mu_1''\}$ and $H = \min \{H_1, H_2)$ we conclude that (38) holds for all $|\mu| \leqq \mu_1$.

Hence there is a unique element $x_t(a, \mu)$ of Σ_a such that

$$x_t(a, \mu) = \mathscr{F}(a, \mu)x_t(a, \mu) \tag{39}$$

Fron the continuity of $\mathscr{F}(a, \mu)$ and from the contracting property it follows that $x_t(a, \mu)$ is continuous on (a, μ) .

Theorem 2

If for some particular (a, μ) fulfilling the requirements of Lemma 5 it happens that $x_t(a, \mu)$, solution of (39), fulfills the relation

$$\mathscr{O}(\Psi^T(0)N(t, x_t(a, \mu), \mu)) = 0 \quad , \tag{40}$$

then $x(a, \mu)$ is a periodic solution of (28) and, conversely, if $\tilde{x}_t(\mu)$, $|\mu| < \mu_1$, is a periodic solution of (28) in Σ_a, then $\tilde{x}_t(\mu) = x_t(a, \mu)$ for some a.

Proof:

The first part is obvious, and the second follows from the fact that $\tilde{x}_t(\mu)$ fulfills (28) for every $t \in R$ and it has to fulfill (30),(31) and (40) according to the properties of $\mathscr{O}$. The results follow from the uniqueness of solution in $\Sigma_{a,H}$ of (30) and (31).

Equation (40) is generally known in the literature as "bifurcation equation" or "determining equation".

Notice that if Λ is empty, i.e., (2) has as only T-periodic solution the identically zero, then there is no relation (40) to fulfill and we conclude that equation (28) has a unique periodic solution $x_t(\mu)$ which depends continuously on μ and tends to 0 as $\mu \to 0$, i.e., $x_t(0) = 0$.

The method to determine T-periodic orbits of (28) for small μ is then to find $x_t(a, \mu)$ corresponding to (30), (31) for $|\mu|$ in some interval $[0, \mu]$, substitue this value in (40) and solve for a in terms of μ.

This method is too difficult to be practical. The main difficulty deriving from the fact that $x_t(a, \mu)$ is generally not known explicitly. On the other hand for any (a, μ) we can find a sequence $x_t^{(k)}(a, \mu)$, of T-periodic function converging uniformly to $x_t(a, \mu)$ due to the fact that it is the fixed point of a contracting mapping.

The sequence is given by:

$$x_t^{(0)}(a, \mu) = \Phi\, e^{Bt} a \tag{41}$$

$$x_t^{(k)}(a, \mu) = \mathscr{F}(a, \mu) x_t^{(k-1)}(a, \mu)$$

Notice that due to the form of $\mathscr{F}(a, \mu)$ we have $x_t(a, 0) = \Phi\, e^{Bt} a$.

If $\mathscr{O}(\Psi^T(0)N(t, x_t(a, \mu), \mu))$ is differentiable with respect to a we can apply the implicit function theorem and decide on the solvability of a as a function of μ in equation (40).

In order to insure this differentiablity we will ask for further restriction on N.

<u>Lemma 6</u>

<u>If</u> $N(t, \varphi, \mu)$ <u>is as in Lemma 5 and moreover</u> $D_\varphi N(t, \varphi, \mu)$ <u>exists and is lipschitzian in</u> φ <u>with Lipschitz coefficient</u> $\bar{\eta}(|\mu|, H)$, $\bar{\eta}$ <u>with the</u>

<u>same properties as in Lemma 5, then the fixed point</u> $x_t(a, \mu)$ <u>of</u> $\mathscr{F}(a, \mu)$ <u>and</u> $\mathscr{O}(\Psi^T(0)N(t, x_t(a, \mu), \mu))$ <u>are differentiable with respect to</u> a <u>for</u> μ <u>and</u> H <u>sufficiently small</u>.

Remarks: The symbol D_φ stands for the Fréchet derivative and $D_a f$, with f a p-vector function, is a $p \times p$ matrix.

Notice that if $N(t, \varphi, \mu) = \mu N^*(t, \varphi)$, with N^* and $D_\varphi N^*$ lipschitzian the conditions of the lemma are fulfilled.

If η, $\bar{\eta}$ do not depend on H the results are valid independently of H.

<u>Proof</u>:

We use induction on the sequence (41). We have $D_a x_t^{(0)}(a, \mu) = \Phi e^{Bt}$. Assuming that $D_a x_t^{(k)}(a, \mu)$ exists we have:

$$D_a y^{(k+1)}(a, \mu)(t) = e^{Bt}(I + \int_0^t D_a\, n(x_t^{(k)}(a, \mu), \mu)(\tau)d\tau -$$

$$- \frac{1}{T}\int_0^T\int_0^t D_a n(x_t^{(k)}(a, \mu), \mu)(\tau)d\tau dt) ,$$

$$D_a x_t^{(k+1)Q}(a, \mu) = (I-U(T))^{-1}\int_t^{t+T} U(t+T-\tau)X_0^Q D_a N(\tau, x_\tau^{(k)}(a, \mu),\mu)d\tau .$$

Here $D_a n(x_t^{(k)}(a, \mu),\mu)$ and $D_a N(\tau, x_\tau^{(k)}(a, \mu),\mu)$ exist due to our hypothesis on N.

Notice that if the mean value of $x_t^{(k)}$ is zero, so is the mean value of $D_a x_t^{(k)}(a, \mu)$.

Hence $D_a x_t^{(k+1)}(a, \mu)$ exists and is continuous. Moreover if $\|e^{Bt}\|_S < M$, we can choose μ small enough as to have $\|D_a x_t^{(k)}(a, \mu)\|_\Sigma < M$ for all k. This can be proved by induction taking into account the Lipschitz property of $D_\varphi N(t, \varphi, \mu)$ in the same way as we proved in Lemma 5 that $\mathscr{F}(a, \mu)$ maps

$\Sigma_{a,H}$ into $\Sigma_{a,H}$.

We check next that $D_a x_t^{(k)}(a, \mu)$ converges uniformly in Σ_a to some function matrix, which is precisely $D_a x_t(a, \mu)$.

Notice that Σ is a complete space and that the sequence of function matrices $D_a x_t^{(k)}(a, \mu)$ is a Cauchy sequence, as we show in the next paragraphs:

(42)
$$\|D_a x_t^{(k+1)}(a, \mu) - D_a x_t^{(k)}(a, \mu)\| \leq$$

$$\leq \bar{\eta}(|\mu|, H)K_1(\|x^{(k)} - x\|_\Sigma + \|x - x^{(k-1)}\|_\Sigma) +$$

$$+ \bar{\eta}(|\mu|, H)K_2\|D_a x^{(k)} - D_a x^{(k-1)}\|.$$

Here we are using as norms of the function matrices the supremum of the norms of its columns considered as vectors

The constant K_1 depends on M and K_2 on the upper bound of $D_\varphi N$ on C_H.

From (38) into (41) it follows that

$$\|x_t^{(k)}(a, \mu) - x_t(a, \mu)\|_\Sigma \leq \frac{\delta_1^k}{1 - \delta_1} \| x_t^{(1)}(a, \mu) - x_t^{(0)}(a, \mu)\|_\Sigma$$

Denote by δ_2 the maximum of $\bar{\eta}(|\mu|, H)K_1$ and $\bar{\eta}(|\mu|, H)K_2$ and choose μ, H small enough to have $\delta_2 < 1$. Let δ be the maximum of δ_1 and δ_2. Then it follows from (42):

$$\|D_a x_t^{(k+1)}(a, \mu) - D_a x_t^{(k)}(a, \mu)\| \leq$$

$$\leq \delta\left(\frac{\delta^k + \delta^{k-1}}{1 - \delta}\right) \| x_t^{(1)}(a, \mu) - x_t^{(0)}(a, \mu)\|_\Sigma +$$

$$+ \delta \| D_a x_t^{(k)}(a, \mu) - D_a x_t^{(k-1)}(a, \mu)\| \leq$$

$$\leq \delta(\Delta^k + \delta\Delta^{k-1} + \dots + \delta^{k-1}\Delta^1) \|x_t^{(1)} - x_t^{(0)}\|_\Sigma +$$

$$+ \delta^k \| D_a x_t^{(1)} - D_a x_t^{(0)}\| \leq$$

$$\leq \delta^k \left(k \frac{1 + \delta}{1 - \delta} + L\right) \|x_t^{(1)} - x_t^{(0)} \|_\Sigma .$$

Here Δ^k stands for $(\delta^k + \delta^{k-1})/(1 - \delta)$, and L is a constant factor relating the norms of $\|x_t^{(1)} - x_t^{(0)}\|_\Sigma$ and $\|D_a x_t^{(1)} - D_a x_t^{(0)}\|$. As we have that $\sum_{k=1}^{\infty} k\delta^k$ converges, it follows that $\{D_a x_t^{(k)}(a, \mu)\}$ is a Cauchy sequence converging to some element of Σ which is $D_a x_t(a, \mu)$.

We are now in condition to state the following theorem which represents the most practical result of the method.

Theorem 3

If N fulfills the conditions required for Lemma 6 besides i),ii),iii) above, with η, $\bar{\eta}$ depending only on $|\mu|$ and if

(43) $$\mathcal{O}(\Psi^T(0)N(t, \Phi e^{Bt} a_0, 0)) = 0$$

$$\det (D_a \mathcal{O}(\Psi^T(0)N(t, \Phi e^{Bt} a_0, 0))) \neq 0 ,$$

then there exists $\mu_1 > 0$ such that equation (28) has a T-periodic solution

$x_t^*(a_o, \mu)$ <u>for</u> $|\mu| < \mu_1$. <u>This solution is continuous in</u> μ <u>and</u> $x_t^*(a_o, 0) = \Phi e^{Bt} a_o$.

Proof:

Notice that $\Phi e^{Bt} a_o = x_t(a_o, 0)$. From the continunity of $x_t(a, \mu)$ with respect to μ it follows, by applying the implicit function theorem to (40), that for $a = a_o$ and $\mu = 0$ we can express a as a function of μ such that $a(0) = a_o$.

The solution $x_t^*(a, \mu)$ is given by $x_t^*(a_o, \mu) = x_t(a(\mu), \mu)$.

Evidently $x_t^*(a_o, 0) = x_t(a_o, 0) = \Phi e^{Bt} a_o$, and this completes the proof.

Remark: The lemma will still be true even if $\eta, \bar{\eta}$ depend on H if $a_o = 0$, since we used the property only to check that $\Phi e^{Bt} a_o = x_t(a_o, 0)$.

Notice also that if H has a factor ϵ we can take it out and consider equations (43) divided by ϵ and we obtain the desired results.

III. The autonomous equation

We will apply here the results of the preceding section to some autonomous equations, in particular to those of the type

$$(44) \qquad \dot{z}(t) = L(z_t) + N(z_t, \mu) ,$$

L and N fulfilling the same conditions of the previous section.

In order to show how the things should be done in the real case we are going to assume that L and N are real functionals over the space $C = C([-r, 0], R^n)$ and we look for real solutions of (44). The complex case is alike but a little simpler because we can diagonalize B and with every eigenvalue we don't need the conjugate to be also an eigenvalue.

In the real case we can always choose Φ (see section I) in such a way that the matrix B is of the form

$$(45) \qquad B = \text{diag}\,(0_q, C_1, \ldots, C_r) ,$$

$$C = \begin{pmatrix} 0 & n_i\omega \\ -n_i\omega & 0 \end{pmatrix} .$$

Here 0_q stands for the $q \times q$ zero matrix, and $n_i\omega$ are the imaginary parts of the elements of Λ, n_i ranging in the positive integers. It may happen that two n_i have the same value for a finite number of indexes.

Contrasting with the nonautonomous case, we cannot expect to preserve the period $T = 2\pi/\omega$ under perturbation. However we do expect that if some periodic solution of (44) tends to some periodic solution of (2) as μ tends to zero, then its period is going to tend to T.

We are going to look for periodic solutions of period $T(\mu) = 2\pi/\omega(\mu)$, with $\omega(\mu) = \omega + \mu\eta$, where we have to determine η in function of μ and the particular solution of (2) to which we approach when μ tends to zero.

With the notation

$$B(\omega(\mu)) = \mathrm{diag}(0_q,\ C_1(\omega(\mu)),\ \ldots, C_r(\omega(\mu))),$$

(46)

$$C_i(\omega(\mu)) = \begin{pmatrix} 0 & n_i(\omega + \mu\eta) \\ -n_i(\omega + \mu\eta) & 0 \end{pmatrix}$$

we write (44) as

$$\dot{w}(t) = B(\omega(0))w(t) + \Psi^T(0)N(\Phi w(t) + z_t^Q,\ \mu)$$

(47)

$$z_t^Q = U(t)z_0^Q + \int_0^t U(t-\tau)X_0^Q N(\Phi w(\tau) + z_\tau^Q,\ \mu)d\tau$$

where $w(t) = (\psi,\ z_t)$.

If we apply the change of variables

(48) $$w(t) = e^{B(\omega(\mu))t}y(t),\quad z_t^Q = x_t^Q,$$

we obtain the systems

$$\dot{y}(t) = \mu\, e^{-B(\omega(\mu))t} B(\eta) e^{B(\omega(\mu))t} y(t) +$$

(49) $$+ e^{-B(\omega(\mu))t} \Psi^T(0) N(\Phi e^{B(\omega(\mu))t} y(t) + x_t^Q, \mu),$$

$$x_t^Q = U(t) x_0^Q + \mu \int_0^t U(t-\tau) X_0^Q N(\Phi e^{B(\omega(\mu))t} y(\tau) + x_\tau, \mu) d\tau \quad ,$$

which is of the form

$$\dot{y}(t) = Ay(t) + F(t, y(t), x_t^Q, \mu, \eta)$$

(50) $$x_t^Q = U(t) x_0^Q + \int_0^t U(t-\tau) X_0^Q\, G(\tau, y(\tau), x_\tau^Q, \mu, \eta) d\tau \;,$$

with $A = 0_q$ and F and G $T(\mu)$-periodic in t.

The functions F and G fulfill all of the conditions which are necessary to apply Lemma 5 and Theorems 2 and 3, even if in this case (50) does not correspond to any single equation like (28). Let us remark again that by x_t we are denoting a functional dependence of elements of C on R and we don't require the existence of $x(t)$ such that $x(t + \theta) = x_t(\theta)$.

If we take

(51) $$f(a, \eta, \mu) = \int_0^{T(\mu)} F(\tau, y(\tau, a, \eta, \mu), x_\tau^Q(a, \eta, \mu), \mu, \eta) d\tau \;,$$

then we obtain that analogously as in Theorem 3

(52) $$f(a_0, \eta_0, 0) = 0 \;, \quad \operatorname{rank} (D_{(a,\eta)} f(a_0, \eta_0, 0)) = p$$

are sufficient conditions to insure the possibility of expressing a and η as functions of μ.

In this case we can determine η and $p-1$ components of a as functions of μ and the other component of a. The arbitrariness of one of the components of a is due to the autonomy of the system, in which a 1-parameter family of periodic solutions corresponds to every closed orbit.

IV. The stability of periodic solutions

The results of section II can also be used to determine the stability characteristics of periodic solutions of functional differential equations.

Consider, for instance, the equation

$$(44)\qquad \dot{x}(t) = L(x_t) + \mu N(x_t, \mu) \ .$$

Let x_t^* be a $T(\mu)$-periodic solution of (44). Take now $z = x - x^*$ and we obtain

$$(53)\qquad \dot{z}(t) = L(z_t) + \mu(N(x_t^* + z_t, \mu) - N(x_t^*, \mu)) =$$

$$= L(z_t) + \mu\, L^*(t, z_t, \mu) + \mu o(|z_t|).$$

Here the linear functional L^* is the Fréchet derivative of $N(x^* + \varphi, \mu)$ with respect to φ and is $T(\mu)$-periodic in t.

Equation (53) gives the behavior of the solutions of (44) with respect to x^*. If we are only interested in what happens in the vicinity of x^* it is sometimes enough to consider the first variational equation

$$(54)\qquad \dot{z}(t) = L(z_t) + \mu L^*(t, z_t, \mu) \ .$$

In the noncritical cases the stability properties of x_t^* can be decided by the knowledge of the characteristic exponents of (54). In fact, if all the characteristic exponents, except one which is zero, have negative real parts, then x_t^* is asymptotically stable with asymptotic phase.

For the general theory of periodic linear functional differential equations see Stokes [5] and Shimanov [6]. For the stability result mentioned above see Stokes [11].

We know that the characteristic multipliers of (54) are continuous in μ and we know their value for $\mu = 0$, namely, they are given by the expontentials of the roots of the characteristic equation (4).

Hence, if x_t^* is going to be at all stable we have to require that there are no roots of (4) with positive real parts. In fact we will require that all the characteristic values of (2) have negative real parts except those in Λ. In order to prove asymptotic stability of x_t^* in this case it is sufficient to show that for μ small enough all the elements of $\Lambda(\mu)$ are in the left hand plane with the exception of one which is at 0.

The decomposition of (54) by Λ yields the following equation for the orbits in $P(\Lambda)$:

$$\dot{w}(t) = Bw(t) + \mu\, \Psi^T(0)L^*(t, \Phi w(t) + z_t^Q , \mu) .$$

Notice now that L^* is $T(\mu)$-periodic and the change of variable $w(t) = e^{B(\omega(\mu))t}y(t)$, $z_t^Q = x_t^Q$ reduces it to the form

$$\dot{y}(t) = \mu(-e^{-B(\omega(\mu))t}B(\eta)e^{B(\omega(\mu))t}y(t) + \tag{55}$$

$$+ e^{-B(\omega(\mu))t}\Psi^T(0)L^*(t, \Phi e^{B(\omega(\mu))t}y(t) + x_t^Q , \mu)).$$

From the work of Stokes and Shimanov we know that corresponding to every characteristic exponent τ there exists a solution $y(t) = e^{\tau t}p(t)$, $x_t^Q = e^{\tau t}\tilde{x}_t^Q$, where $p(t) = p(t + T(\mu))$. (Just like in the case with no lag).

Substituting this value of $y(t)$ in (55) and taking $\tau = \mu\nu$ we obtain the following equation for $p(t)$:

$$\dot{p}(t) = -\mu\nu p(t) + \mu(-e^{-B(\omega(\mu))t}B(\eta)e^{B(\omega(\mu))t}p(t) +$$

(56)

$$+ e^{-B(\omega(\mu))t}\Psi^T(0)L^*(t, \Phi e^{B(\omega(\mu))t}p(t) + \tilde{x}_t^Q, \mu)) .$$

This equation is of the type studied in section II, and we can find, by means of the determinimg equations, what are the values of ν for which we have $T(\mu)$-periodic solutions of (56). These values are the characteristic exponents and x_t^* is asymptotically stable if all but one have negative real parts.

In most cases we don't know what x_t^* is exactly, but we know its limit value when μ tends to zero, and this value is in general good enough to determine the stability conditions for small values of μ.

V. <u>An example</u>

Consider the equation

$$\dddot{z}(t) + a\ddot{z}(t) + b^2\dot{z}(t) + kz(t-r) + \mu\psi(z(t-r)) = 0 , \tag{57}$$

in which a, b^2, k, r and μ are positive constants and ψ is a real function of the real variable z such that, for any initial φ in C there is a unique solution of (57) with initial value φ at zero for all positive t.

Equation (57) arises from a control system with a nonlinearity and a delay of value r in the feedback.

For some values of the parameters and a special form of ψ we are going to determine the periodic solutions of (57) which tend to periodic solutions of

$$\dddot{v}(t) + a\ddot{v}(t) + b^2\dot{v}(t) + kv(t-r) = 0 \tag{58}$$

as μ tends to zero.

The characteristic equation of (58) is given by

$$\lambda^3 + a\lambda^2 + b^2\lambda + ke^{-r\lambda} = 0 \tag{59}$$

Using procedures similar to the ones used in Chapter 13 of [12] we find that for $r=2$, $a = (64-\pi)/8\pi$, $b=1$ and $k = a\pi^2\sqrt{2}/64$, equation (59) has exactly two purely imaginary roots $\pm i\omega$, $\omega = \pi/8$, and that the rest of the roots have negative real parts. (For the details see [13].) This means that

$P(\Lambda)$ is a plane in C where all the periodic orbits of (58) are contained.

We can write (58) as

(60) $$\dot{u}(t) = \int_{-r}^{0} d\eta(\theta)u(t+\theta),$$ u a vector with components

u_1, u_2, u_3 and

$$\eta(\theta) = \begin{pmatrix} 0 & u(\theta) & 0 \\ 0 & 0 & u(\theta) \\ -ku(\theta+r) & -b^2u(\theta) & -au(\theta) \end{pmatrix},$$

where

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geqq 0 \end{cases}.$$

The matrix B and Φ are given, respectively by

$$B = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \quad \text{and}$$

$$\Phi = \begin{pmatrix} \cos\omega\theta & \sin\omega\theta \\ -\omega\sin\omega\theta & \omega\cos\omega\theta \\ -\omega^2\cos\omega\theta & -\omega^2\sin\omega\theta \end{pmatrix}, \quad \theta \in [-r, 0].$$

The value of $\Psi^T(0)$ turns out to be

$$\Psi^T(0) = \frac{1}{\delta}\begin{pmatrix} \beta(1-\omega^2)+a\gamma\omega & a\beta+\gamma\omega & \beta \\ \gamma(1-\omega^2)-a\alpha\omega & ar-\alpha\omega & \gamma \end{pmatrix},$$

where $\alpha \doteq 0.84$, $\beta \doteq -0.30$, $\gamma \doteq 1.60$ and $\delta \doteq 2.25$.

We write now equation (57) as

$$\text{(61)} \qquad \dot{x}(t) = \int_{-r}^{0} d\eta(\theta)x(t+\theta) + f(x_t) \quad , \qquad \eta(\theta) \text{ as}$$

above, x a vector with components x_1, x_2, x_3 and

$$f(x_t) = -\mu \begin{pmatrix} 0 \\ 0 \\ \psi(x_1(t-2)) \end{pmatrix}$$

With the decomposition

$$x_t = \Phi y(t) + x_t^Q \quad , \qquad y(t) = (\Psi, x_t) \quad ,$$

we obtain the equation

$$\text{(62)} \qquad \dot{y}(t) = By(t) + \Psi^T(0)f(\Phi y(t) + x_t^Q) \; .$$

After the substitutions are made we obtain

$$\text{(63)} \qquad \begin{aligned} \dot{y}_1(t) &= \frac{\pi}{8}\, y^2(t) - \mu\frac{\beta}{\delta}\, \psi(-y_2)(t) + (x_t^Q(-2))_1) \\ \dot{y}_2(t) &= \frac{\pi}{8}\, y_1(t) - \mu\,\frac{\gamma}{\delta}\, \psi(-y_2)(t) + (x_t^Q(-2))_1) \; . \end{aligned}$$

These equations are in the form (47) and we can apply the procedure explained there. We are going to take $\psi(x) = x - x^3$ in our example.

We apply the transformation (48) with

$$e^{B(\omega(\mu))t} = \begin{pmatrix} \cos\omega\, t & \sin\omega\, t \\ -\sin\omega\, t & \cos\omega\, t \end{pmatrix} .$$

We obtain for (52) with $a_o = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ the following:

$$f(a_o, \eta_o, 0) = \begin{pmatrix} -\eta a_2 \\ \eta a_1 \end{pmatrix} - \frac{1}{\delta} \begin{pmatrix} -\frac{\gamma}{2} a_1 + \frac{\beta}{2} a_2 \\ -\frac{\beta}{2} a_1 - \frac{\gamma}{2} a_2 \end{pmatrix} +$$

$$+ \frac{1}{\delta} \begin{pmatrix} -\frac{3}{8}\gamma a_1^3 + \frac{3}{8}\beta a_2^3 + \frac{3}{8}\beta a_1^2 a_2 - \frac{3}{8}\gamma a_1 a_2^2 \\ \frac{3}{8}\beta a_1^3 - \frac{3}{8}\gamma a_2^3 - \frac{3}{8}\gamma a_1^2 a_2 - \frac{3}{8}\beta a_1 a_2^2 \end{pmatrix} = 0 .$$

By taking $a_2 = 0$, which we can do due to the arbitrariness of one of the components of a_o, this equation reduces to

$$\frac{1}{2}\gamma a_1 - \frac{3}{8}\gamma a_1^3 = 0$$

$$\delta\eta a_1 + \frac{1}{2}\beta a_1 - \frac{3}{8}\beta a_1^3 = 0 ,$$

which yields $a_1 = 0$, η undetermined and $a_1^2 = \frac{4}{3}$ with $\eta = 0$.

This means that our equation has two periodic solutions (letting aside the phase) tending respectively to 0 and to the solution of $\dot{u} = Bu$ with "radius" $\sqrt{4/3}$ as μ tends to 0.

We apply now the procedure of the previous section to compute approximately the characteristic exponents of the first variational equation (54).

We take $x_t^* = \Phi\, e^{Bt} a_o$, $B = B(\omega)$, and we have

$$y_2(t) = (\Phi(-2)e^{Bt}a_o)_2 = a_1 \sin \omega\, t.$$

The value of $L^*(t, z_t)$ is given by

$$L^*(t, \mathbf{z}_t) = \begin{pmatrix} 0 \\ 0 \\ -\psi'(a_1 \sin \omega\, t)\ \mathbf{z}_1(t-2) \end{pmatrix} .$$

Decomposing $\mathbf{z}_t$ by Λ in order to have $z_t = \Phi\, w(t) + z_t^Q$ and performing the change of coordinates $w(t) = e^{B(\omega(\mu))t}y(t)$, $z_t^Q = x_t^Q$, we obtain equation (56) with

$$e^{-B(\omega(\mu))t}\Psi^T(0)L^*(t,\ \Phi e^{B(\omega(\mu))t}p(t)) =$$

$$= \frac{1}{8}\,\psi'(a_1 \sin \omega\, t)(-\sin \omega(\mu)t\ \cos \omega(\mu)t)p(t) \times$$

$$\times \begin{pmatrix} \beta \cos \omega(\mu)t - \gamma \sin \omega(\mu)t \\ \beta \sin \omega(\mu)t + \gamma \cos \omega(\mu)t \end{pmatrix} .$$

As $\psi'(a_1 \sin \omega t) = 1 - 3a_1^2 \sin^2 \omega t$ we obtain the determining equations

$$\begin{pmatrix} -\kappa + \frac{1}{\delta}(\frac{1}{2}\gamma - \frac{9}{8}\gamma a_1^2) & -\eta + \frac{1}{\delta}(\frac{1}{2}\beta - \frac{3}{8}\beta a_1^2) \\ \eta + \frac{1}{\delta}(-\frac{1}{2}\beta + \frac{9}{8}\beta a_1^2) & -\kappa + \frac{1}{\delta}(\frac{1}{2}\gamma - \frac{3}{8}\gamma a_1^2) \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = 0 .$$

As $a_1^2 = \frac{4}{3}$ and $\eta = 0$, we have κ given by the eigenvalues of the matrix

$$\begin{pmatrix} -\gamma & 0 \\ \beta & 0 \end{pmatrix} ,$$

which are 0 and $-\gamma$.

As $\gamma > 0$ we conclude that our solution is asymptotically stable.

VI. Higher order approximations and example.

Consider again the equation

$$(28) \qquad \dot{x}(t) = L(x_t) + N(t, x_t, \mu) ,$$

where N fulfills conditions i), ii) and iii) and moreover admits a large enough number of derivatives with respect to its arguments.

By means of the successive approximations given by (41) we can obtain in some cases the coefficients of the lower order terms in the expansion of

$$(64) \qquad \mathscr{O}(\Psi^T(0)N(t, x_t(a, \mu), \mu) = F(a, \mu)$$

in terms of a and μ. Here (64) is the determining function for (28) and $x_t(a, \mu)$ represents the unique fixed point defined by (39).

We will show now how the knowledge of these terms may help us in determing the existence and the order of magnitude of a perioic solution of (28). This method may work even in the case in which the application of Theorem 3 has failed because $\det(D_a F(a_o, 0)) = 0$.

To simplify notation we will consider the scalar case with $a_o = 0$, i.e. we assume $F(0, 0) = 0$.

Suppose also that by means of (41) we have been able to obtain the lowest order terms in a and μ for $F(a, \mu)$. By this we mean that we can write

(67) $$F(a, \mu) = \mu^{\nu}(k_o a^{m_o} + k_1 a^{m_1}\mu^{n_1} + \dots + k_p a^{m_p}\mu^{n_p}) + f(a, \mu) = \mu^{\nu} P(a,\mu) + f(a,\mu),$$

where $P(a, \mu)$ has been chosen in such a way that we take into account only the terms lying in the side of steepest slope of the Newton polygon, i.e., the terms for which $\nu n_j/(m_o - m_j)$ is a minimum. Let

(66) $$\lambda = \frac{n_j}{m_o - m_j}, \qquad j = 0, \dots, p \ .$$

If we now substitute

(67) $$a = \bar{a}\, \mu^{\lambda}$$

we obtain

$$\bar{F}(\bar{a}, \mu) = \mu^{\nu + \lambda m_o} (k_o \bar{a}^{m_o} + \dots + k_p \bar{a}^{m_p}) + f(\bar{a}, \mu) =$$

$$= \mu^{\nu + \lambda m_o} \bar{P}(\bar{a}) + f(\bar{a}, \mu).$$

where $f(a, \mu)$ is $o(\mu^{\nu+\lambda m_o})$ for a fixed $\bar{a}$.

If we want to find $\bar{a}(\mu)$ for μ sufficiently small such that $\bar{F}(\bar{a}(\mu), \mu) = 0$ we apply the implicit function theorem. Owing to the form of $\bar{F}(\bar{a}, \mu)$ what we have to do is solve for $\bar{a}$ in

(68) $\qquad \overline{P}(\overline{a}) = 0$ and check $\frac{\partial}{\partial \overline{a}} \overline{P}(\overline{a}) \neq 0$

at these values.

If we find such a value of $\overline{a}$ we get, by using (67), that there exists a solution of (28) which tends to 0 like

(69) $\qquad a(\mu) = \overline{a}\, \mu^{\lambda}$

as μ tends to zero.

In the case in which $F(a_o, 0) = 0$ for a_o different from zero the treatment is analogous, but expanding in terms of $a - a_o$. The same will apply for periodic solutions with amplitudes tending to ∞ when μ tends to zero. This corresponds to the case of negative λ. It can be treated by expanding in terms of the recipocal of a.

We present now an example due to J. K. Hale in which the above technique is utilized.

Consider

(66) $\qquad \dot{x}(t) = (\frac{\pi}{2} + \mu)x(t-1)(1-x^2(t))$

The unperturbed equation and its adjoint are given by

$$\dot{u}(t) = \frac{\pi}{2}\, u(t-1) \quad \text{and}$$
$$\dot{v}(s) = \frac{\pi}{2}\, v(s+1)$$

The bases for the generalized eigenvalues Φ and Ψ can be chosen as:

$$\Phi = (\varphi_1, \varphi_2),\ \varphi_1(\theta) = \sin \frac{\pi}{2}\, \theta,\ \varphi_2(\theta) = \cos \frac{\pi}{2}\, \theta,\ \theta\, \epsilon[-1, 0]$$

$$\Psi^T = \frac{2}{\nu^2}\begin{pmatrix} \psi_1 - \frac{\pi}{2}\psi_2 \\ \\ \frac{\pi}{2}\psi_1 + \psi_2 \end{pmatrix}$$

$$\nu^2 = \frac{1}{1+\frac{\pi^2}{4}} \;, \quad \psi_1(\theta) = \sin\frac{\pi}{2}\theta,\ \psi_2(\theta) = \cos\frac{\pi}{2}\theta,\ \theta \in [0,\ 1]$$

This choice has been made in order to have $(\Psi,\ \Phi) = I$, where here

$$(\psi,\ \varphi) = \psi(0)\varphi(0) - \frac{\pi}{2}\int_{-1}^{0} \psi(\xi+1)\varphi(\xi)d\xi$$

Equation (66) can then be written, by using

$x_t = \Phi y(t) + x_t^Q$, as

$$(67) \qquad \begin{cases} y = By + \Psi^T(0)N(x_t,\ \mu) \\ x_t^Q = U(t)\varphi^Q + \int_0^t U(t-\tau)X_0^Q\, N(x_\tau,\ \mu)d\tau \end{cases} ,$$

where

$$B = \begin{pmatrix} 0 & -\pi/2 \\ \pi/2 & 0 \end{pmatrix}, \quad \Psi^T(0) = \begin{pmatrix} -\pi/\nu^2 \\ 2/\nu^2 \end{pmatrix}$$

$$N(x_t,\ \mu) = -\mu x(t-1)(1-x^2(t)) + \frac{\pi}{2}x(t-1)x^2(t) .$$

Let now $\omega(\mu) = -\frac{\pi}{2} + \mu\beta$ and

$$B(\omega(\mu)) = \begin{pmatrix} 0 & -\frac{\pi}{2} + \mu\beta \\ \\ -\frac{\pi}{2} - \mu\beta & 0 \end{pmatrix} .$$

We preform the change of variables

$y = e^{B(\omega(\mu))t}z$ in (67) and we obtain:

$$\dot{z}(t) = -\mu e^{-B(\omega(\mu))t}B(\beta)e^{B(\omega(\mu))t}z(t) + e^{-B(\omega(\mu))t}\Psi^T(0)N(x_t, \mu) \; . \tag{68}$$

Here we have

$$N(x_t, \mu) = \mu(z_1 \cos \omega t + z_2 \sin \omega t + x_t^Q(-1))(1-(-z_1 \sin \omega t + z_2 \cos \omega t + x_t^Q(0))^2) +$$
$$+ \frac{\pi}{2} (-z_1 \cos \omega t - z_2 \sin \omega t + x_t^Q (-1))(-z_1 \sin \omega t + z_2 \cos \omega t + x_t^Q(0))^2.$$

As the system is autonomous we can altogether forget about z_2, say, and we obtain for a vector with components $(a, 0)$ and for β the determining equations for $\mu = 0$

$$\frac{1}{\nu^2} \begin{pmatrix} \frac{\pi^2}{16} a^3 \\ -\frac{\pi}{8} a^3 \end{pmatrix} = 0$$

The only solutions is $a = 0$, but for this value the Jacobian with respect to a and β vanishes.

We look then for the lowest order terms.

In our determining equations we have terms like μa, μx_t^Q, a^3, $a^2(x_t^Q)^2$ $a(x_t^Q)^3$, etc.

We check first the order of x_t^Q. If x_t is periodic we have the representation

$$x_t^Q = \int_{-\infty}^{t} U(t-\tau)X_0^Q \, N(x_\tau, \mu)d\tau$$

As $N(x_\tau, \mu)$ has μa as its lowest order term it turns out that this is the order of x_t^Q. This means that the only terms to be considered are μa and a^3.

Taking these into account we obtain the determining equations:

$$\frac{\pi^2}{\nu^2\, 16}\; a^3 - \frac{\pi}{2\, \nu^2}\; a\mu = 0$$

$$\mu\beta a - \frac{\pi}{\nu^2\, 8}\; a^3 + \frac{1}{\nu^2}\; a\mu = 0$$

Hence

$$a = \sqrt{\frac{8\mu}{\pi}} \quad , \quad \beta = 0$$

We have for the jacobian with respect to a and β:

$$\begin{vmatrix} -\frac{3\pi^2 a^2}{16\nu^2} - \frac{\pi\mu}{2\nu^2} & \mu\,\beta - \frac{3\pi a^2}{8\nu^2} + \frac{\mu}{\nu^2} \\ 0 & \mu a \end{vmatrix}$$

which differs from zero for the value obtained for a.

We have then a solution close to

$$(69) \qquad x_t = \Phi e^{B(-\frac{\pi}{2})t} \begin{pmatrix} a \\ o \end{pmatrix} \; , \quad a = \sqrt{\frac{8\mu}{\pi}} \qquad .$$

If we substitute x by $\sqrt{\mu}x$ in equations (66) and we get a problem which is solvable in the first approximation:

$$(70) \qquad x(t) = -\left(\frac{\pi}{2} + \mu\right)x(t-1)(1 - \mu x^2(t))$$

The bifurcation equations turn out to be

$$\begin{pmatrix} 0 \\ \beta a \end{pmatrix} + \frac{1}{\nu^2} \begin{pmatrix} \frac{\pi^2}{16} a^3 - \frac{\pi}{2} a \\ -\frac{\pi}{8} a^3 + a \end{pmatrix} = 0$$

Hence $a = 0$ is a solution, the same as $a = \sqrt{\frac{8}{\pi}}$. For this last value the jacobian differs from zero and this proves that for μ small enough there is a periodic solution of (70) tending to (69) with $a = \sqrt{\frac{8}{\pi}}$, or, what it is the same, a solution of (66) tending to (69).

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Carlos Perelló
Center for Dynamical Systems
Division of Applied Mathematics
Brown University